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A $\mathbf{Z}_2$ Classification for 2D Fermion Level Crossing

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Abstract

We demonstrate that the number of fermionic zero modes of the static 2-dimensional Dirac operator in the background of $SU(2)$ static gauge-Higgs field configurations is a topological invariant modulo four. Static configurations which are everywhere odd under parity with even-parity pure gauge behaviour at infinity admit $4n$, $n \in \mathbf{Z}$, zero modes of the Jackiw-Rebbi (JR) type. Odd-parity configurations with odd-parity pure gauge behaviour at infinity are topologically disconnected from the vacuum and admit $4n + 2$ fermionic zero energy solutions. The classification implies the collapse of half of the fermion zero modes upon embedding a 2-dimensional gauge-Higgs configuration (string) with odd-parity pure gauge behaviour at infinity into the 3-dimensional Minkowski space.

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1 Introduction

Anomalous fermion level crossing in gauge theories is intrinsically connected to their topologically nontrivial periodic vacuum structure. For theories with the gauge symmetry spontaneously broken there appears a potential barrier between the topologically inequivalent vacua which is proportional to the mass scale of the theory. At finite temperatures thermally induced transitions above the barrier are accompanied by fermion level crossing. It is due to the presence of static gauge-higgs fields with Chern-Simons number $CS = 1/2$, the sphalerons, which are saddle points to the field equations. In the context of the standard $SU(2) \times U(1)$ electroweak theory such configurations induce an odd number of normalizable zero energy solutions to the 3D Dirac equation.

It was previously found [1, 2] that for the gauge fields A_i having odd parity everywhere the Chern-Simons number is a topological charge. In a gauge where A_i is odd under parity transformation the U-odd configurations (where U is defined via the asymptotic behaviour of A_i at infinity, $A_i \to -i\partial_i U(x)U(x)^{-1}$) have $CS = n+1/2$, $n \in \mathbf{Z}$, and admit an odd number of fermionic zero modes. They contribute to unsuppressed $B+L$ violating thermal transitions. The U-even configurations have $CS = n$ with an even number of fermionic zero modes. They are homotopically equivalent to the vacuum and are not directly connected to the U-odd configurations. In other words the $3D$ Dirac operator in the background of odd-parity gauge-Higgs fields admits a topological invariant number modulo 2 of zero modes. The presence of U-odd configurations is a consequence of a nontrivial Z_2 structure in the space of static finite energy configurations in the electroweak theory. It follows from the existence of the homotopy groups $[S^2/\mathbf{Z}_2, SU(2)/\mathbf{Z}_2] = \mathbf{Z}_2$ and $[S^3/\mathbf{Z}_2, SU(2)/\mathbf{Z}_2] = \mathbf{Z} \times \mathbf{Z}_2$ [2].

In the present note we examine the eigenvalue spectrum of the static two dimensional Dirac operator in the background of $SU(2)$ and $U(1)$ gauge fields respectively. The presence of zero modes in the background of embedded string loops of the Nielsen-Olesen type [3] was first demonstrated by Jackiw and Rebbi [4]. For Z-string loops the issue was studied by Earnshaw and Perkins [5].

The organization of the paper is the following. In section 2 we discuss the $SU(2)$ theory in (2+1) dimensions and the problem of extension of its topological defects into (3+1) dimensions. In section 3 we discuss the $U(1)$ gauge theories in different dimensions and the problem of embedding the low dimensional topological defects into higher dimensions. We close by summarizing the results of the paper.

2 SU(2) gauge theory in (2+1) dimensions

We discuss the spectrum of the Dirac operator in the background of a static configuration.

To that end we consider the 2D Dirac equation for the Dirac 2D fermion in an

external gauge field $A_i(x)$

$$\sigma_i \nabla_i \psi = \sigma_i(\partial_i - iA_i(x))\psi(x) = i\lambda\psi(x), \tag{1}$$

where $i = 1, 2$. The matrix σ_3 plays the role of the 4D γ_5 since it anticommutes with all 2D σ matrices. Thus the Dirac spinor can be split into two one component chiral spinors $\psi = (\psi_1, \psi_2)$. It is worth emphasizing that a chirality cannot be defined in 3D. The anticommutation property of the σ_3 matrix automatically means that in 2D for any spinor ψ which obeys eq.(1) one can construct the spinor $i\sigma_3\psi$ which corresponds to the eigenvalue $-i\lambda$. We thus prove that in 2D all non-zero eigenvalues are paired $(i\lambda, -i\lambda)$. Moreover for the case of an $SU(2)$ gauge group we can choose the eigenfunction for any eigenvalue λ to be "real"

$$i\sigma_2 i\tau_2 \psi^*(x) = \psi(x), \tag{2}$$

while the real eigenfunction for $-i\lambda$ reads $i\sigma_3\psi(x)$. Here τ_i, $i = 1, 2, 3$ are the isospin matrices.

It is easy to see that there is also a doubling of zero modes of the Dirac operator. Indeed let us consider $(\psi_1, 0)$ and $(0, \psi_2)$ as a basis of the kernel of the Dirac operator. For these components we have the following equations

$$\nabla\psi_2 = 0, \quad \bar{\nabla}\psi_1 = 0. \tag{3}$$

Here $\nabla = \nabla_1 - i\nabla_2$ and $\bar{\nabla} = \nabla_1 + i\nabla_2$. It is now easy to see that for any $\psi_1 \neq 0$, $\bar{\nabla}\psi_1 = 0$, one can explicitely construct $\psi_2 = i\tau_2\bar{\psi}_1 \neq 0$, $\nabla\psi_2 = 0$, where τ_2 is the isospin matrix. This proves the doubling of zero modes.

In the case of an odd-parity gauge field, $A(x) = -A(-x)$, we have an additional degeneracy for all the non-zero eigenvalues. Let ψ be a real eigenfunction for an non-zero eigenvalue $i\lambda$. We can easily check that the real spinor $i\sigma_3\psi(-x)$ corresponds to the eigenvalue $i\lambda$ while $\psi(-x)$ corresponds to $-i\lambda$. Moreover the eigenfunctions constructed above are linearly independent. Indeed let us assume that the wave functions ψ and $i\sigma_3\psi(-x)$ were linearly dependent, i.e. $\psi(x) = a\sigma_3\psi(-x)$, where a is a constant. Then it is easy to see $a = \pm 1$. Let us now consider the reality condition of ψ, eq.(2). By combining it with the condition of linear dependence one can get that $\psi(x) = \pm i\sigma_3\psi(-x)$. It clearly contradicts with $a = \pm 1$, if $\psi \neq 0$. We thus have that each eigenfunction with non-zero λ generates a different eigenfunction for the same $i\lambda$. Hence each non-zero eigenvalue is twice degenerate, and moreover there is a pairing of levels $(i\lambda, -i\lambda)$. This doubling for non-zero modes is of course a result of the odd parity behaviour of the gauge field.

Let us consider the zero modes of the above Dirac operator in an odd-parity gauge field. If there is a zero mode we can then construct the above four wavefunctions which obey the same equation. However such a wave function can be odd or even with respect to a parity transformation $\psi(-x) = \pm\psi(x)$. Therefore the number of linearly independent zero modes may be less than 4. However it is easy to see that the number of linearly independent zero modes is always even. Indeed for any

real zero mode with a wave function $\psi(x)$ we can separately construct a linearly independent one $i\sigma_3\psi(x)$ (as discussed above).

Thus we see that the number of zero modes of the Dirac operator is invariant modulo 4 (but not modulo 2 as we had in 3 dimensions) when we smoothly change the external odd-parity gauge field. Indeed if we change smoothly the external (odd under parity) gauge field a non-zero eigenvalue $i\lambda$ can cross zero. At some moment $i\lambda$ gets to be zero while by continuity the four above eigenfunctions being linearly independent (the wave functions for different values of λ are orthogonal) are zero mode wave functions. Hence under smooth deformations of the gauge field (odd under parity) the number of zero modes can change only by $4k$, where k is an integer.

It is worth emphasizing that the quadroupling of the non-zero modes is of course a result of the odd parity behaviour of the gauge field as well as of the above index theorem modulo 4. In the case of non-constrained gauge fields (without a definite parity) there is no index theorem since the number of zero modes is always even while the non-zero modes are paired and there is no additional doubling for them. This corresponds to the trivilatity of the homotopy group $\pi_1(SU(2)) = 0$.

Thus at least for the gauge group $SU(2)$ the topological classes of parity odd gauge fields are two classes which allow for the number of fermionic zero modes to be 2 mod 4 and 0 mod 4, respectively. Following the approach of ref. [1, 2] one can now show that these classes correspond to the odd-parity gauge fields with even and odd pure gauge behaviour at ininity, $A_i(x) \to -i(\partial_i U U^{-1})(x)$, $U(x) = \pm U(-x)$. Since a vacuum configurationt obviously does not allow for fermionic zero modes and has $U(x) = U(-x)$ behaviour at infinity we conclude that the sector of gauge fields with such a behaviour requires 0 (modulo 4) fermionic zero modes. Thus the homotopically non-trivial sector corresponds to gauge fields with $U(x) = -U(-x)$. These fields require 2 (modulo 4) fermionic zero modes.

The above group element $U(x)$ is a map from S^1 into $SU(2)$. Thus we have proved that there is a non-trivial homotopy group [6]

$$[S^1/\mathbf{Z}_2, SU(2)/\mathbf{Z}_2] = \mathbf{Z}_2. \tag{4}$$

This group is responsible for the above index theorem. Formally eq. (4) follows from the facts that $S^1/\mathbf{Z}_2$ is isomorphic to S^1 and $SU(2)/\mathbf{Z}_2 = SO(3)$.

We will now discuss an embedding of two-dimensional topologically non-trivial objects, such as strings, into three dimensional space. In particular we consider the 2D Dirac operator in the external field of the Nielsen-Olesen string, i.e. we consider a 2D slice perpendicular to the line of the vortex. The fermionic zero mode in such a field is proportional to $1/r$ at infinity (r is the distance to the line of the vortex). Hence the normalization integral is logarithmically divergent. Nevertheless these zero modes are still relevant if we assume that the 2D space is compactified. For example the simplest way to compactify the space is to introduce an external metric which makes the normalization integral convergent. In this case we have a couple of zero modes [4] that agrees with the above analysis.

We now construct a loop of such a string. Since any configuration in three dimensional space is contractable such a loop may allow for any number of fermionic zero modes. Therefore we have to formulate the prescription for making a three dimensional object if we want to find a correspondence between the homotopy properties of two and three dimensional configuration. We shall assume that the radius of the loop is much larger than the size of its core. We will also assume that we construct a three dimensional gauge field configuration which is odd under parity transformation. In this case there are two different situations as it has been demonstrated in ref. [1, 2]. One of two sets of three dimensional configurations allows for an even number of fermionic zero modes. In this case one considers a twisting [7] of the string which corresponds to pure gauge behaviour of the three dimensional gauge field $A_i(x) \to -i(\partial_i U U^{-1})(x)$ at infinity with even $U(x) = U(-x)$. The second set of three dimensional fields corresponds to a non-trivial twisting of the string. The resulting three dimensional gauge fields have pure gauge behaviour at infinity with odd $U(x) = -U(-x)$. These fields allow for an odd number of fermionic zero modes.

From the analysis of [1, 2] it follows that some fermionic zero modes which existed on the two-dimensional slice do not cause an appearance of zero modes of the three dimensional Dirac operator in the string loop configuration. Therefore we conjecture that a non-trivial twisting of the string configuration (odd-parity pure gauge behaviour at infinity) kills half of the zero modes of the Dirac operator since a three dimensional Dirac operator has an odd number of normalizable zero modes in this case. The reason for the disappearance of half of the zero modes is probably that only one of their linear combinations can satisfy the condition of single-valuedness of the wavefunction when we extend the two dimensional object into three dimensions. In the opposite case of a topologically trivial twist (even-parity pure gauge behaviour at infinity) all zero modes might survive.

We thus conjecture that there is a correspondence of the $\mathbf{Z}_2$ structure of the theory in (2+1) dimensions to that of the (3+1) dimensional theory. This correspondence is non-trivial because it depends on the twisting. For an appropriate twisting the non-trivial homotopy class in (2+1) dimensions maps to the non-trivial homotopy class in (3+1).

3 U(1) gauge theory in (3+1), (2+1) and (1+1) dimensions

Here we shall consider the gauge theories with $U(1)$ gauge group.

Let us first start with a gauge theory in (1+1) dimensions defined on $I \times \mathbf{R}$, where the interval $I = [-\pi, \pi]$ stands for the "space". In this case the homotopic properties of static configurations are described by the fact that the homotopy group of maps $[\partial I, U(1)] = 0$, where ∂I consists of the two end points of the interval I. Therefore there is no index theorem for the static Dirac operator in this case. Let us now turn to the static gauge fields even under the parity transformation. This restriction is due

to the fact that the sphaleron configuration in (1+1) dimensions is a constant gauge field (see e.g. [8]). In this case we have the following non-trivial homotopy group $[\partial I/\mathbf{Z}_2, U(1)/\mathbf{Z}_2] = \mathbf{Z}_2$. Therefore there are two classes of gauge fields which are not continuously related to each other: with odd-parity and even-parity pure gauge behaviour at the ends of the space interval $I = [-\pi, \pi]$. A particular representative of the non-trivial homotopy class is just the above sphaleron $A_x = 1/2$. More generally $A_x = n/2$, where n is an odd-integer. For the trivial class we have n to be even. The existence of a non-trivial homotopy group $[\partial I/\mathbf{Z}_2, U(1)/\mathbf{Z}_2] = \mathbf{Z}_2$ implies the existence of an index theorem for the Dirac operator in 1 dimension in the presence of even-parity gauge fields. The Dirac operator has one zero mode for odd n and 0 zero modes for even n (we assume the antiperiodicity of the fermionic wavefunctions).

For the $U(1)$ gauge theory in (2+1) the relevant homotopic group formally reads

$$[S^1/\mathbf{Z}_2, U(1)/\mathbf{Z}_2] = \mathbf{Z} \tag{5}$$

since $U(1)/\mathbf{Z}_2 = \mathrm{R}P^1 = U(1)$. If we denote an element of the $U(1)$ group as $\exp i\phi(\theta)$, where θ is a coordinate on S^1, $\theta \in [0, 2\pi]$, then the classes of the above homotopic group are represented by group elements for which $\exp 2i\phi(\theta)$ is periodic on $[0, \pi]$. This group element can have any integer winding number on the interval $[0, \pi]$. This is the same winding number as for the element $\exp i\phi(\theta)$ on the interval $[0, 2\pi]$. Therefore all the classes of $\mathbf{Z}$ are split by parity into two subgroups: with odd and even winding numbers, respectively.

Let us now consider the spectrum of the Dirac operator. In the case of a gauge group $U(1)$ the eigenfunctions cannot be generally chosen real. For a generic gauge field there is no doubling of zero modes and the number of zero modes can be both odd and even in contrast to the above example. However there is a pairing of non-zero wave functions given by $\psi(x)$ and $i\sigma_3\psi(x)$ for eigenvalues λ and $-\lambda$ respectively. We see that the number of zero modes can change only by an even number under smooth deformations of gauge fields due to a pairing of non-zero levels. Hence we have an index theorem that the number of fermionic zero modes is invariant modulo 2. Actually in this case it is well known that the index of the 2D Dirac operator is given by the winding number of the gauge field [6].

Let us now consider the case of an odd under parity gauge field. Because of non-reality the wave functions $\psi(x)$ and $i\sigma_3\psi(-x)$ which correspond to the same eigenvalue λ can be linearly dependent. Therefore in general there is no doubling of non-zero modes. The same is obviously true for zero modes. Indeed, given a wave function $\psi(x)$ for a zero mode we can construct three more wave functions $\psi(-x)$, $i\sigma_3\psi(x)$ and $i\sigma_3\psi(-x)$. It is easy to see that the condition of linear dependence of all these 4 wave functions is not contradictory. Therefore we see that when we constrain the gauge fields to be odd under parity the index theorem mod 2 for zero modes still holds without any changes. This fact agrees with the above statement that no homotopy classes appear due to oddness under parity. Instead we get a classification of the usual $\mathbf{Z}$ classes: with odd and even winding numbers. The sense of the index theorem can be expressed as follows: it is not possible to change the

number of zero modes from odd to even and back under a smooth deformation of the gauge field.

We conclude that the number of zero modes of the Dirac operator is invariant modulo 2. Thus it is odd in the non-trivial class of the gauge fields and even in the trivial class with respect to the above splitting of $\mathbf{Z}$ into $2\mathbf{Z}$ and $2\mathbf{Z}+1$. Such a conclusion agrees with the index theorem for the Dirac operator which is coupled to a $U(1)$ gauge field, which states that the difference of numbers of left-handed and right-handed zero modes is given by a winding number of the gauge field.

Let us now consider an embedding of (1+1) static configurations into (2+1) dimensions. Under such an embedding the coordinate on an interval I is naturally identified with S^1 which corresponds to the polar angle θ in two dimensional space. Then it is easy to see that odd fields $A_x = (n+1/2)$ cannot be embedded since they would correspond to a not single-valued group element at infinity. Hence only gauge fields $A_x = n$ can be embedded. Thus the spectrum of the Dirac operator reduces under such an embedding since half of the zero modes of the 1-dimensional Dirac operator become non-single-valued. The above $\mathbf{Z} \to 2\mathbf{Z} \oplus 2\mathbf{Z}+1$ homotopic classification in (2+1) dimensions corresponds to the odd-parity gauge fields with even or odd pure gauge behaviour at infinity. Thus there is no connection between the $\mathbf{Z}_2$ structure in (1+1) and (2+1) dimensions under such an embedding.

We now turn to the 3D $U(1)$ gauge theory. Since the group $\pi_2(U(1)) = 0$ there is no index theorem for the Dirac operator. We firstly consider the sector of odd-parity gauge fields. The relevant homotopy group for this case is also trivial $[\mathrm{R}P^2, U(1)] = 0$. This follows from the fact that [6]

$$[RP^2, U(1)] = \mathrm{Hom}(\mathbf{Z}_2, \mathbf{Z}) = 0. \tag{6}$$

We also present here a simple intuitive explanation of this result. The $\mathrm{R}P^2$ sphere is equivalent to a hemisphere with identified opposite points on the boundary. Formally by following what we did in the $SU(2)$ case we get two classes of gauge fields which are odd under parity: the ones with odd and even respectively pure gauge behaviour at infinity, i.e. on S^2. These $U(1)$ group elements are given by exponentials of imaginary functions $\exp i\alpha(x)$. We want now to show that the class of odd $g = \exp i\alpha(x)$ is empty. Let us consider the boundary of the above hemisphere: g is odd on this boundary which is S^1. Then by continuity it is clear that $\alpha(x)$ is shifted by 2π when moving around this boundary. Due to the non-triviality of $\pi_1(U(1)) = \mathbf{Z}$ this group element which of course belongs to a non-trivial class of $\pi_1(U(1))$ is not contractable to unity. Hence when extended to the whole hemisphere there appears a singularity at some point on it. Therefore we conclude that the class of odd g is empty because by definition we have to consider smooth group elements on S^2. Such an analysis could not prove the absence of odd classes for the $SU(2)$ group because $\pi_1(SU(2)) = 0$.

Thus there is no non-trivial classes in the 3D case for the group $U(1)$.

Let us now consider the case of an $SU(2) \times U(1)$ gauge group in (2+1) and (3+1) dimensions. In (2+1) dimensions there is a non-trivial $\mathbf{Z}_2$ structure which

depends on the embedding of the $\mathbf{Z}_2$ group into $SU(2) \times U(1)$. In particular there is a non-trivial $\mathbf{Z}_2$ structure if $\mathbf{Z}_2 \in U(1)$. However as it was shown in ref. [2] the corresponding homotopy group $[\mathrm{R}P^2, SU(2) \times U(1)/\mathbf{Z}_2] = 0$ for the case $\mathbf{Z}_2 \in U(1)$. Therefore this part of the $\mathbf{Z}_2$ structure of a (2+1) theory is lost under the embedding into (3+1) theory. Therefore only the $\mathbf{Z}_2$ structure that corresponds to the center of $SU(2)$ survives under this embedding.

4 Conlusions

We extended our previous study of homotopic classification of odd-parity deformed sphalerons to lower dimensions. Remarkably enough we found for the $SU(2)$ gauge theory a non-trivial $\mathbf{Z}_2$ structure in (2+1) dimensions which implies an index theorem modulo 4 for the Dirac operator. This implies that the number of zero modes of the Dirac operator is a topological invariant which takes values (0 or 2) modulo 4. This corresponds to the existence of the non-trivial homotopy group $[S^1/\mathbf{Z}_2, SO(3)] = \mathbf{Z}_2$.

We argued for the existence of a non-trivial correspondence of such a $\mathbf{Z}_2$ structure for static gauge fields in two and three dimensions. More specifically the embedding of a 2-dimensional static configuration, such as a Nielsen-Olesen vortex, into a 3-dimensional one, such as a string loop results in the collapse of an odd number of zero modes from the 2 mod 4 it may otherwise possess.

We also investigated the $\mathbf{Z}_2$ structure in the $U(1)$ gauge theories in (1+1), (2+1) and (3+1) dimensions. We argued that there is no non-trivial map of the $\mathbf{Z}_2$ structure from the (2+1) dimensional theory into the (3+1) dimensional one, and also there is no non-trivial map of the $\mathbf{Z}_2$ structure from the (1+1) dimensional theory into the (2+1) dimensional one.

For its potential physical implications the first part of our results is the more relevant. It suggests that the existence of zero modes of the Dirac operator in 2 dimensions does not imply the existence of zero modes in three dimensions. Thus we see that the B-violating properties of three dimensional configurations, such as the electroweak strings, are not directly related to the homotopic properties of the two dimensional configurations which are embedded into the three dimensions.

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